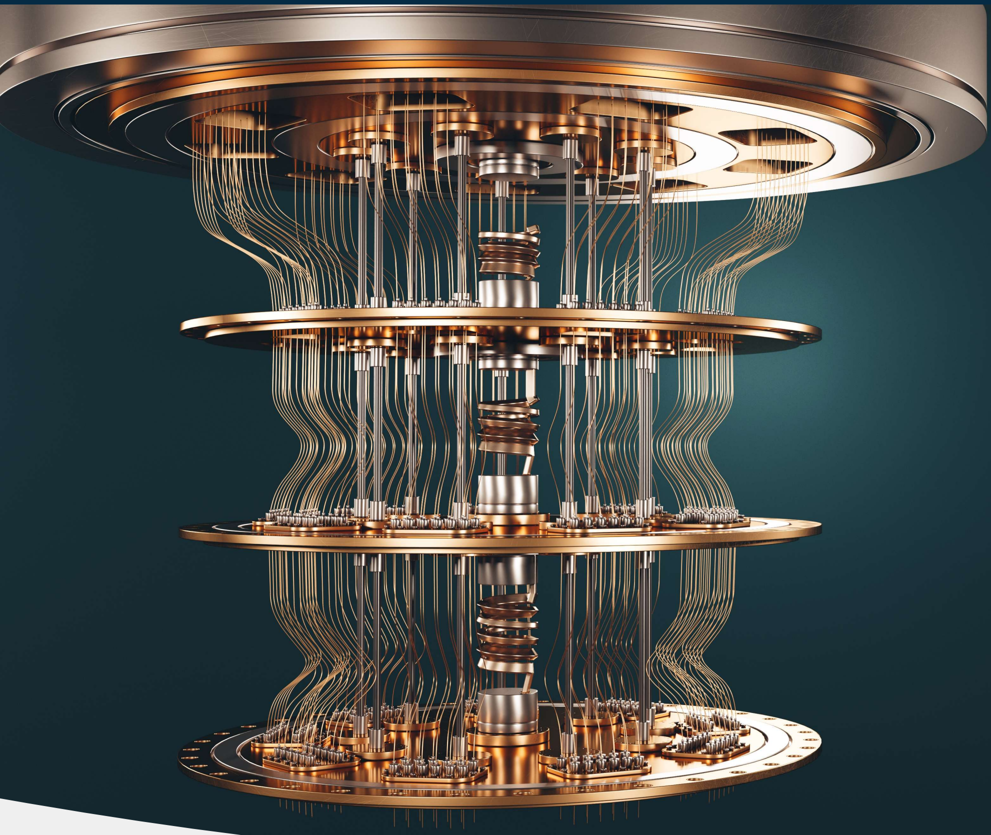
Think openly, build securely



White paper:

Quantum Computing Threat: An Overview of Post-Quantum Cryptography

PQShield

Updated August 2022

Contents

Executive Summary

In July 2022, NIST announced the first selection of standards for post-quantum cryptography [NIS16]. Whilst there exist a large corpus of literature on post-quantum cryptography, the field as a whole has not yet coalesced into a fully formalized discipline. As a result, even people acquainted with it may struggle to get a clear understanding of the technical principles and how they are put into practice.

The goal of this document is to provide the reader with an understanding of the key technical ideas used in post-quantum cryptography. As such, it is a rather technical document. However, we focus here on the high-level principles, and try to avoid low-level details when possible. An interesting fact is that subfields of post-quantum cryptography (lattice-based, code-based, multivariate, etc.) often share several techniques and design principles, even though they may work on very different mathematical objects. We highlighted these connections when we felt they were relevant.

The announced NIST standards are primarily based on lattice-based and hash-based cryptography, which are only a subset of the fields discussed here. Round 4 of the NIST competition will open up the field a little to include other families such as code-based and isogeny-based cryptography, but the announced call for new proposals [NIS16] is designed to look for schemes from all of the fields in this document and beyond - so it will be interesting to see all the submissions, This document will therefore be fully updated once the candidates become known, to encompass all the latest research in this exciting field.

1 Classical Public-Key Cryptography

Cryptography deals with the issue of ensuring secure communications over insecure channels, and does so via mathematical methods.

When two parties share a secret key, symmetric-key cryptography provides efficient ways of ensuring the confidentiality (via symmetric encryption, for example AES [AES01]) and integrity of communications (via message authentication codes, for example HMAC [HMA08]).

However, these solutions are inapplicable when the parties *do not* share a secret key in advance. Public-key cryptography was invented precisely to deal with these situations.

In this section, we first describe some hard problems, the building blocks upon which public-key cryptography is based today. We then explain the key technical ideas, and show how cryptographic protocols put them into practice. Finally, we explain how quantum computers jeopardize the security provided by current public-key cryptography.

1.1 Hard problems

Computationally hard problems, or hard problems for short, is a broad notion encompassing problems that require a significant (ideally, intractable) amount of resources to be solved. Cryptography makes a peculiar use of hard problems; other fields often try to avoid these problems, but cryptography uses them as the foundation of secure schemes. This is typically done by establishing an equivalence between the security of a scheme and the intractability of a hard problem.

Until recently, two hard problems (or variants thereof) have been ubiquitous in public-key cryptography: *integer factorisation*, and the *discrete logarithm problem*.

Factorisation-related problems

The prime factorisation problem is one of the simplest, and probably best-known, hard problems in cryptography.

The prime factorisation problem

Let p and q be two prime integers and $N = p \times q$. Given N, find p and q.

Whether this problem is actually hard (with the computational power currently available) depends on the set from which p and q are picked. For actual cryptosystems, p and q are picked from a set large enough so that prime factorisation is infeasible in practice.

While a few cryptosystems [Rab79, Wil84] rely solely on prime factorisation, many more of them [GM82, Pai99] rely on related problems, such as the RSA problem.

The RSA problem

Let p and q be prime integers, $N = p \times q$, and e and d two integers such that

$$d \times e = 1 \bmod (p-1) \times (q-1)$$

Given N, e and $m^e \bmod N$ for a random $0 \leq m < N$, find m.

The hardness of the RSA problem is the assumption underlying the security of the eponymous encryption and signature schemes by Rivest, Shamir and Adleman [RSA78].

The prime factorisation problem is *at least as hard* as the RSA problem, however whether they are equivalent is still an open question. In practice, schemes based on the RSA problem choose their parameters to make the related factorisation problem hard.

Problems related to the discrete logarithm

Another class of hard problems heavily relied upon by cryptography relates to the discrete logarithm in finite cyclic groups. A finite group is a finite set with some added algebraic structure: for example, the set $\mathbb{Z}_q$ of integers modulo q is a finite group. We say that a finite group $G = \langle g \rangle$ is cyclic if it is generated by an element g. We first consider the discrete logarithm problem, or DLOG.

DLOG - discrete logarithm problem

Let $G = \langle g \rangle$ be a finite cyclic group. Given g and g^a, find a.

Similarly to prime factorisation and RSA, while some schemes rely solely on DLOG, others rely on two related problems: DDH and CDH, first used by Diffie and Hellman [DH76].

DDH and CDH - Diffie-Hellman problems

Let $G = \langle g \rangle$ be a finite group. Given g, g^a, g^b for a, b random:
Decision (DDH): Distinguish (g^a, g^b, g^{ab}) from a triple (g^a, g^b, g^c) with c random.
Search (CDH): Compute g^{ab}.

DLOG is at least as hard as CDH, which is at least as hard as DDH. In other words:

$$DLOG \geq CDH \geq DDH.$$

Assessing equivalence in general is still complicated; there are groups in which CDH is as hard as DLOG, and there are groups for which DDH is easy but CDH seems hard.

An active subfield of public-key cryptography is elliptic-curve cryptography. An elliptic curve is essentially the set of points (x, y) verifying a certain equation for fixed a, b, p:

$$y^2 = x^3 + ax + b \bmod p.$$

Taking G to be an elliptic curve often allows to design very compact schemes based on the problems DLOG, CDH, DDH, etc.

1.2 Encryption, key exchange and key encapsulation

Encryption schemes, key-exchange protocols and key encapsulation mechanisms are three related protocols which solve the same problem: establishing a secure communication between two parties over an insecure channel.

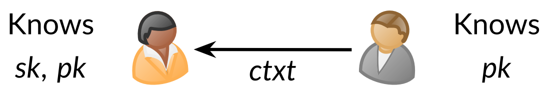
- A key-exchange protocol is a (possibly interactive) protocol at the end of which two parties agree on a shared symmetric key.
- A (public-key) encryption scheme is a scheme with public and private keys, where a public key *pk* allows to encrypt a message *msg*, and the corresponding private key *sk* allows to recover *msg*.
- A key encapsulation mechanism (or KEM) also has public and private keys, a public key *pk* allows to encrypt a random symmetric key K (*encapsulation*), and the associated private key *sk* allows to recover K (*decapsulation*).

Most basic constructions achieve either encryption schemes [RSA78, ElG85] or key-exchange protocols [DH76] naturally, but there are simple generic conversions that transform any protocol of one type into a protocol of an other type.

Encryption

For simplicity, each time we describe a scheme, Alice will denote the owner of the private key *sk* (for decryption) and Bob the owner of the corresponding public key *pk* (for encryption).

In an encryption scheme, Alice () generates a key pair (sk, pk), keeps the private key sk to herself, and publicly distributes the public key pk. Bob () computes a ciphertext $ctxt = \mathrm{Enc}(pk, msg)$.



Upon reception of $ctxt$, Alice applies a decryption algorithm to it and recovers the message: $msg = \mathrm{Dec}(sk, ctxt)$. Alice and Bob now both know a secret msg.

Given $\mathrm{Enc}(pk, msg)$, it must be hard to recover msg, except if we know the private key sk, which allows to invert $\mathrm{Enc}(pk, \cdot)$. We have:

$$\mathrm{Dec}(sk, \mathrm{Enc}(pk, msg)) = msg.$$

This description is arguably very generic. We illustrate how to apply this idea in practice with a simplified version of the RSA cryptosystem, described hereafter. The idea is that while exponentiating a message ($ctxt = msg^e \bmod N$) is easy, "inverting" this operation to recover msg is hard if the RSA problem is hard.

RSA encryption

- The private key sk is a couple (p, q) of distinct primes and a value d;
- The public key pk is the product $N = p \times q$, and a value e such that
  $$e \times d = 1 \bmod (p-1) \times (q-1);$$
- To encrypt a message msg, compute:
  $$ctxt = msg^e \bmod N$$
- To decrypt $ctxt$, compute:
  $$msg = ctxt^d \bmod N$$

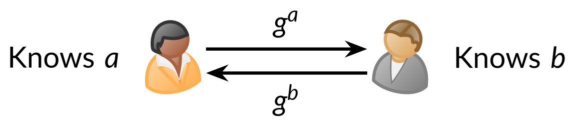
A "magic" mathematical property of d and e is that, for any element x in $\mathbb{Z}_N$:

$$(x^e)^d = x^{ed} = x \bmod N.$$

This allows Alice (who knows d) to recover msg. Of course, this simplistic description conceals a number of details which have to be addressed in real applications. This is the purpose of transforms such as RSA-OAEP [BR95].

Diffie-Hellman key exchange

The Diffie-Hellman protocol is a key-exchange protocol. Given a random (generating) element g of a group G, Alice chooses a random a, computes g^a, keeps a to herself and send g^a to Bob. Similarly, Bob sends some g^b to Alice. Upon reception of g^b, Alice raises it to the power of her secret exponent a, and obtains $(g^b)^a$. Bob does the same, and obtains $(g^b)^a$.



Thanks to the properties of the exponentiation, both obtain the same value:

$$(g^a)^b = (g^b)^a = g^{ab}$$

For someone observing the exchange, finding g^{ab} is as hard as solving CDH, which is conjectured to be hard.

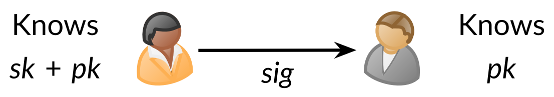
This protocol verifies some nice additional properties: it is *static* (Alice can reuse her secret value a for an exchange with another party), and *non-interactive* (both parties can send their g^x before receiving an answer). We will later see that post-quantum protocols struggle to achieve both properties at once.

Chosen-ciphertext security

Most basic encryption schemes are only proven secure against chosen-plaintext attacks (CPA), where an attacker must encrypt honestly. In chosen-ciphertext attacks (CCA), an attacker can craft malicious ciphertexts. These can be mitigated by applying generic conversions, the most famous being the Fujisaki-Okamoto transforms [FO99a, FO99b].

1.3 Signatures

Digital signatures solve the same problem as their physical counterparts: proving the authenticity of a document. (sk, pk) will denote a pair of signing and verification keys, and H a cryptographic hash function. Alice attests the validity of a document to Bob by appending a digital signature $sig \leftarrow \text{Sign}(sk, msg)$ to it. sig guarantees that Alice is the rightful author of the document, and detects any modification.



The hash-then-sign paradigm

The Hash-then-sign paradigm is arguably the most "intuitive" approach for building digital signatures. The message is first hashed by a hash function H into a challenge $\mathbf{c}$. A signature is a value sig such that $f_{pk}(sig) = \mathbf{c}$ for some public function f_{pk} parameterized by pk. f_{pk} is a trapdoor one-way function, which means it is easy to compute and hard to invert (*one-way*), except if we know sk (the *trapdoor*).

With the RSA problem, this idea is easily put in application, as described below. In practice, executing this idea securely requires adddi-tional tweaks described in RSA-PSS [BR98].

RSA signature

- The private key sk is a couple (p, q) of distinct primes and a value d;
- The public key pk is the product $N = p \times q$, and a value e such that

$$e \times d = 1 \bmod (p - 1) \times (q - 1);$$

- To sign a message msg, compute:

$$sig = H(msg)^d \bmod N$$

- A signature is accepted if and only if:

$$H(msg) = sig^e \bmod N$$

The Fiat-Shamir paradigm

The philosophy of the Fiat-Shamir paradigm [FS87] is very different from Hash-then-sign: it is based on *identification protocols*. These protocols are *interactive* protocols that allow a prover to prove its identity to a verifier. Most of them follow a *commit-challenge-response* communication flow illustrated in Figure 1.

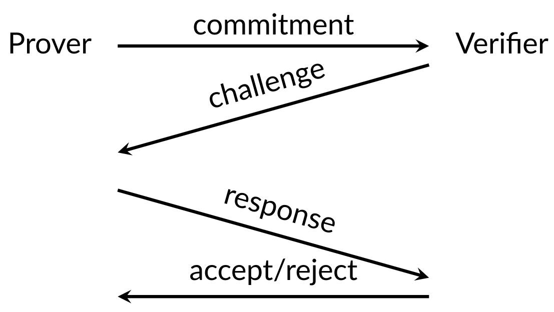


Figure 1: A 3-pass identification protocol

The most important phase is the response, where the prover solves a problem dependent on both the commitment and the challenge, and which would be hard to solve without their private key. The Fiat-Shamir paradigm turns an identification protocol into a signature scheme by making the challenge a hash of both the commitment and the message. Of course, the resulting scheme is now non-interactive.

Schnorr signature

- The private key sk is a value x;
- The public key pk is $h = g^x$;
- To sign a message msg:
  - **Commit:** Select a random y, and compute $u = g^y$;
  - **Challenge:** Compute $c = H(u, msg)$;
  - **Response:** z = y - cx;

  The signature is $sig = (u, z)$.
- A signature is accepted if and only if:

$$u = g^z h^{H(u,msg)}.$$

An illustration of this principle is the Schnorr signature scheme [Sch90] described above, where each of Alice's signatures is a proof that she knows the discrete logarithm of the public key.

1.4 The impact of quantum computers

Most of the cryptographic schemes based on the prime factorisation, RSA, discrete logarithm and Diffie-Hellman problems would be assumed secure if not for quantum computers.

In 1994, Shor [Sho94] showed that these classically hard problems would be *easy* to solve on a large scale quantum computer. Since quantum computers were mostly theoretical objects at the time, there was no immediate impact.

However, there has been significant progress on building these computers, driven by large companies (Microsoft, IBM, etc.) and state actors (China, USA, EU), and the prospect of a practical quantum computer becomes more tangible every year.

This has led the cryptographic community, the industry and many standards bodies to plan a replacement of today's widely used public-key cryptography by a quantum-safe alternative: *post-quantum cryptography*.

Summary

Inception:	1976
Hard Problems:	RSA, Factoring, CDH, DDH
Enc/KEM:	(elliptic-curve) Diffie-Hellman, RSA encryption
Signatures:	Schnorr, RSA signatures

2 Lattice-based Cryptography

A lattice is a set generated by integer linear combinations of the columns of a matrix. Thus, lattice-based cryptosystems and hard problems typically involve matrices. Lattice-based cryptography is rather recent (1996) compared to the other subfields, but it has seen a steady growth since its inception.

With a few exceptions, lattice-based cryptography started with very theoretical constructions targeting provable security. As it stands, several schemes are proven secure under the hardness of various lattice problems. However, not all proofs are equal, in the sense that some proofs have a limited practical relevance [CKMS16].

Today, there exist several cryptographic constructions based on lattices. Beyond encryption and signatures, more advanced constructions have been proposed, such as homomorphic encryption, identity-based encryption, etc.

The efficiency of cryptographic schemes based on generic lattices is moderate. Many schemes rely on more structured lattices, and achieve high efficiency in the process. In the initial set of standards by NIST [NIS22], three out of the four selected schemes are based on structured lattices: Kyber [SAB+20], Dilithium [LDK+20] and Falcon [PFH+20].

2.1 Hard problems

There is a myriad of conjectured hard problems in lattice-based cryptography. The most common are SIS and LWE. Both work with matrices having their entries in a finite ring $\mathcal{R}$ (for example $\mathbb{Z}_q$ or $\mathbb{Z}_q[x]/(x^d+1)$).

SIS - short integer solution

Given $\mathbf{A} \in \mathcal{R}^{n\times m}$, find a short vector $\mathbf{v} \neq 0$ such that $\mathbf{Av} = 0$.

Solving SIS without the shortness constraint is straightforward linear algebra. However, forcing the solution to be short adds a geometric constraint which makes this problem much harder.

Another widespread hard problem is *Learning With Errors*, or LWE for short. The definition of LWE gives cryptosystems' designers the freedom to choose not only $\mathcal{R}, m, n$, but also the secret distribution, error distribution, etc. The secret and error distributions (which are public) typically have a small support.

LWE - learning with errors

Let $\mathbf{A} \in \mathcal{R}^{n\times m}$ be a uniformly random and $\mathbf{b} = \mathbf{A}^t\mathbf{s} + \mathbf{e}$, where $\mathbf{s} \in \mathcal{R}^n$ and $\mathbf{e} \in \mathcal{R}^m$ are vectors sampled from the *'secret' distribution* and *'error' distribution*, respectively.

Decision: Distinguish $(\mathbf{A}, \mathbf{b})$ from values sampled uniformly.

Search: Given $(\mathbf{A}, \mathbf{b})$, find $\mathbf{s}$.

This versatility is a double-edged sword. Along with the rich algebraic structure of LWE, it allows to build several constructions on LWE beyond signatures and key agreement, the most pre-eminent being *homomorphic encryption*, which enables secure computation over encrypted data. However, this also leaves room for cryptosystems' designers to unwittingly use insecure parameters.

The most prevalent flavors of LWE are standard LWE, *module*-LWE (or MLWE) and *ring*-LWE (or RLWE). Standard LWE takes $\mathcal{R} = \mathbb{Z}_q$, the integers modulo q. MLWE takes

$\mathcal{R}$ to be some polynomial ring, for example $\mathcal{R} = \mathbb{Z}_q[x]/(x^d + 1)$. RLWE is the special case of MLWE where $n = 1$. MLWE and RLWE introduce some structure which can be exploited to have more efficient schemes (faster and more compact). However, this added structure also means that the underlying hardness assumptions are more aggressive. SIS also exists in similar flavors.

As an illustration, RLWE with one sample posits that:

$$(a, a * s + e) \in \mathcal{R}^2$$

is hard to distinguish from uniform, where a, s, and e are sampled from some uniform, secret and error distributions over $\mathcal{R}$, respectively. Because of its simplicity, we will take RLWE to illustrate many of our examples.

2.2 Key exchange and encryption

2.2.1 Encryption

RLWE encryption

- The private key contains a, s, e;
- The public key is $(a, b = a * s + e)$;
- The encryption of *msg* is *ctxt* $= (u, v)$, with:

$$u = r * a + e_1$$
$$v = r * b + e_2 + \left\lceil \frac{q}{2} \right\rfloor msg$$

  where e_1, e_2 are error vectors.
- To decrypt a ciphertext *ctxt*, the owner of the private key computes:

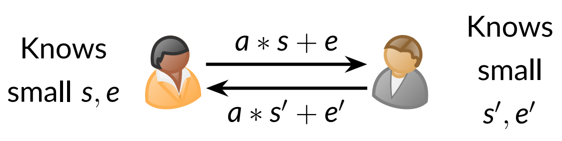
$$v - u * s,$$

  which is equal to $\left\lceil \frac{q}{2} \right\rfloor$ *msg* plus some noise. As the noise is small (thus concentrated in the low bits) and *msg* is encoded in the high bits, it is easy to recover *msg*.

There are several approaches for building encryption schemes using lattices. The box "RLWE Encryption" above presents a simplified version (using RLWE) of the most common approach [LPR10, LP11]. The selected standard Kyber [SAB+20] is based on this approach.

2.2.2 Noisy Diffie-Hellman

One could imagine a naive adaptation of the Diffie-Hellman key-exchange using RLWE. Here, with a being a public element:



However, this would give only an *approximately* shared secret since:

$$a * s * s' + s' * e \neq a * s * s' + s * e'.$$

To cope with this issue, the notion of *reconciliation* has been introduced [DXL12, Pei14]. The idea is that one of the parties sends a *hint* to the other party, so that they agree on the same shared secret. This is known as *noisy Diffie-Hellman*.

Unfortunately, this approach lacks a few advantages of the classical Diffie-Hellman. It is interactive and cannot be used with static keys (e.g. Alice cannot use s, e twice). A more mundane issue is the presence of a patent on reconciliation [Din12].

2.3 Signatures

Secure digital signatures using lattices have not been immediate to obtain. The first secure proposals (for both the hash-then-sign and Fiat-Shamir paradigms) were designed in 2008.

2.3.1 The GPV framework

Early attempts [GGH97, HHP+03] to adapt the hash-then-sign paradigm to the lattice setting took the public key *pk* and private key *sk* to be, respectively, a long basis and a short basis of a same lattice. The signing procedure does the following:

1. Hash the message into a point $H(msg)$ of the ambient space of the lattice.
2. Use *sk* to find a lattice point *v* close to $H(msg)$.

The verification procedure checks that *v* is close to $H(msg)$, and uses *pk* to check that it is also in the lattice. As performing step **2.** is hard to do with a short basis but easy with a long basis, it was expected that breaking this signature scheme would not be easier in practice than solving hard lattice problems. This idea is illustrated in Figure 2; one can see that the signature lies at the intersection of the lattice and of a parallelepiped generated by the private key and centered at $H(msg)$, which makes it unique.

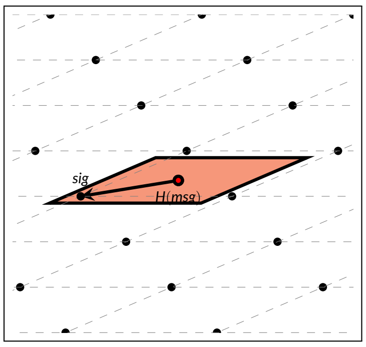


Figure 2: Hash-then-sign *à la* [GGH97, HPS98]. Signatures leak the private key.

However, if step **2.** is not done properly, each signature will lie in a parallelepiped having the form of the private key, and as such would leak a little bit of information about its geometry. This was exploited in devastating attacks [NR06, DN12]; in which a few thousand (msg, sig) pairs leaked enough information to fully recover the private key.

A countermeasure was proposed by Gentry, Peikert and Vailuntanathan in 2008 [GPV08]. The idea is to use a randomized variant of the algorithm used in step **2.** of the original idea [GGH97, HPS98]. This randomization ensures that one message has several valid signatures, and is done in a way which eliminates any correlation between the secret key and the distribution of the signatures. In addition to thwarting the attacks of [GGH97, HPS98], this actually makes the framework of [GPV08] provably secure under the hardness of standard lattice problems.

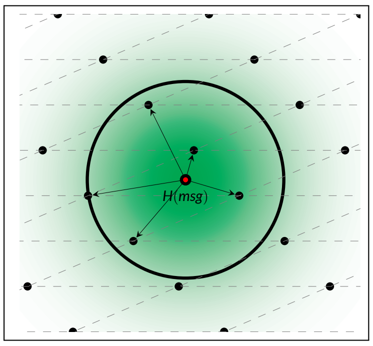


Figure 3: Hash-then-sign *à la* [GPV08]. Signatures no longer leak the private key.

The selected standard Falcon [PFH+20] is an application of the GPV framework. This technique can also be used to build more advanced primitives such as (hierarchical) identity-based encryption [GPV08, CHKP10, DLP14], attribute-based encryption [Boy13, BGG+14], etc.

2.3.2 Fiat-Shamir with aborts

As with their hash-then-sign counterparts, initial attempts [HPS01] to build lattice signatures with the Fiat-Shamir paradigm were quickly cryptanalysed [GJSS01, GS02]. Like for hash-then-sign schemes, each signature leaked a small part of the private key.

A provably secure mitigation was proposed by Lyubashevsky [Lyu09]. The issue with previous attempts was that the underlying protocol lacked the *zero-knowledge* property, and the point of failure was in the *response* step of the protocol. In the attacked schemes, this step induced a distribution of the signatures correlated to the private key, hence the attacks.

This weakness did not appear for classical Fiat-Shamir schemes.

Lyubashevsky observed that randomly aborting (and starting over) the protocol, in a carefully chosen way, allowed to *eliminate* the correlation of the signatures with the private key and nullified this class of attacks. This addition enabled the zero-knowledge property, which was the only property missing in order to make the lattice-based schemes based on Fiat-Shamir provably secure. This technique, by Lyubashevsky, is called *Fiat-Shamir with Aborts*.

Fiat-Shamir with aborts (using LWE)

- The private key sk is two matrices $\mathbf{S}, \mathbf{E}$ of small norm.
- The public key is $(\mathbf{A}, \mathbf{T} = \mathbf{A} \times \mathbf{S} + \mathbf{E})$.
- To sign a message msg:
  - **Commit:** Generate small random vectors $\mathbf{y}_1, \mathbf{y}_2$ and compute the commitment $\mathbf{u} = \mathbf{A} \times \mathbf{y}_1 + \mathbf{y}_2$;
  - **Challenge:** Compute $\mathbf{c} = H(\mathbf{u}, msg)$;
  - **Response:** Compute $\mathbf{z}_1 = \mathbf{y}_1 + \mathbf{S} \times \mathbf{c}$ and $\mathbf{z}_2 = \mathbf{y}_2 + \mathbf{E} \times \mathbf{c}$;
  - **Abort:** With a certain probability $p(sig, sk)$, restart;

  The signature is $sig = (\mathbf{c}, \mathbf{z}_1, \mathbf{z}_2)$.
- A signature is accepted if and only if:
  - $(\mathbf{z}_1, \mathbf{z}_2)$ is short;
  - $\mathbf{c} = H(\mathbf{A} \times \mathbf{z}_1 + \mathbf{z}_2 - \mathbf{T} \times \mathbf{c}, msg)$.

One can see that this scheme is surprisingly similar to Schnorr signatures described in Section 1.3: the public element $a \in G$ is now a matrix $\mathbf{A}$, the challenge $u = g^y$ is replaced with $\mathbf{u} = \mathbf{A} \times \mathbf{y}_1 + \mathbf{y}_2$, and the underlying hard problem DLOG has been replaced with LWE.

We note one important addition; the *abort* step is added as to avoid any leakage of the key and make the scheme secure.

The selected standard Dilithium [LDK+20] is based on Fiat-Shamir with aborts.

Summary

Inception:	1996
Assumptions:	LWE (*Learning with Errors*), SIS (*Short Integer Solution*), NTRU
Enc/KEM:	FrodoKEM, Kyber, Saber
Signatures:	Dilithium, Falcon, qTESLA

3 Code-based Cryptography

Error-correcting codes usually serve to guarantee the integrity and reliability of communication over unreliable channels, by detecting and removing errors. Code-based cryptography uses them in a completely different way, by deliberately adding errors to the point that removing them is hard, except for someone who knows a secret description of the code. Since it mostly entails simple algebraic operations (Gaussian elimination, multiplication, sometimes inversion) on finite field elements, code-based cryptography is often amenable to fast hardware implementations.

Code-based cryptography was first introduced by McEliece [McE78] in his eponymous encryption scheme. His original scheme remains fundamentally secure, is reasonably fast and has short ciphertexts, but a very large public key. Many attempts have been made at making it more efficient, but doing so in a secure manner has proven to be delicate.

Achieving secure code-based signatures has been an even more difficult task. Novel proposals in this direction are being made, but only the test of time will determine if these efforts are successful. Code-based schemes BIKE [ABB+20], HQC [AAB+20] and Classic McEliece [ABC+20] have been selected as Round 4 candidates and are still being considered for standardization by NIST [NIS22].

3.1 Problems

Code-based cryptography relies on linear error-correcting codes (or codes for short). These are generated by matrices over finite fields (for simplicity, this exposition will focus on the field $\mathbb{F}_2$, the integers modulo 2). The generating matrix of a code C is often denoted $\mathbf{G}$, and we often associate to it a parity-check matrix of C, denoted by $\mathbf{H}$ and which verifies $\mathbf{G} \times \mathbf{H}^t = 0$.

Syndrome decoding

The most common code-based problem is the syndrome decoding problem.

Syndrome decoding

Given a matrix $\mathbf{H} \in \mathbb{F}_2^{k\times n}$ and a *syndrome* $\mathbf{s} \in \mathbb{F}_2^k$, find $\mathbf{e} \in \mathbb{F}_2^n$ of Hamming weight at most t such that $\mathbf{H} \times \mathbf{e} = \mathbf{s}$.

The syndrome decoding problem is very similar to the SIS problem described in Section 2.1: the matrix $\mathbf{A} \in \mathcal{R}^{n\times m}$ is replaced by $\mathbf{H} \in \mathbb{F}_2^{k\times n}$, and the constraint on the norm of the solution is replaced by a constraint on its Hamming weight. In a certain parameter regime (not used in cryptography), syndrome decoding is a NP-complete problem.

One can come up with variations of the syndrome decoding problem. One such variation is the rank syndrome decoding problem, which replaces $\mathbb{F}_2$ by a different field, and imposes a condition on the *rank* of the solution instead of its Hamming weight. The rank metric family of schemes [AAB+19, ABD+19] are based on this variation. Another popular modification is to impose a (quasi-)cyclic structure, like schemes based on QC-MDPC codes do.

Code indistinguishability

Another hard problem, or rather a family of hard problems, informally states that for a given family $\mathcal{C} = \{C_i\}_i$ of codes, it is hard to distinguish a random matrix $\mathbf{G}$ generating a code $C = C_i$ from a random matrix.

At a high level, it is somewhat similar to the EIP problem described in Section 4.1. The similarity does not stop here; while this problem is presumed hard for some families of codes (e.g. Goppa, QC-MDPC and LRPC codes), many other choices (Reed-Solomon, LDPC, etc.) have led to insecure schemes.

3.2 Encryption

The idea underlying code-based encryption schemes is that a message will be encrypted by adding some noise to it, and that removing this noise is hard except if one knows a secret description of some code.

McEliece

McEliece encryption

- The private key contains a permutation **P**, an invertible matrix **S**, and a matrix **G** generating a linear code C capable of correcting t errors. The algorithm for correcting errors using **G** is public
- The public key is the product $\hat{\mathbf{G}} = \mathbf{SGP}$.
- The encryption of a message *msg* is:

  $$ctxt = msg \times \hat{\mathbf{G}} + \mathbf{z},$$

  with **z** a vector of Hamming weight t.
- To decrypt a ciphertext *ctxt*, the owner of the private key computes:

  $$ctxt \times \mathbf{P}^{-1} = msg \times \mathbf{SG} + \mathbf{z} \times \mathbf{P}^{-1}.$$

  Since the error vector $\mathbf{z} \times \mathbf{P}^{-1}$ has weight t, the matrix **G** can be used to decode the above value to $msg \times \mathbf{S}$, and then multiplying this by $\mathbf{S}^{-1}$ yields the plaintext *msg*.

A common way of using codes for encryption comes from McEliece's original scheme, described above. The hardness of syndrome decoding and of distinguishing $\hat{\mathbf{G}}$ from a random matrix is necessary for McEliece's encryption scheme to be secure.

McEliece's original proposal has large public keys, since these are large matrices. The use of structured matrices allows to diminish this size, though this automatically implies a more aggressive hardness assumption.

Niederreiter

McEliece's cryptosystem admits a *dual* version which was proposed by Niederreiter in 1986 [Nie86]. It replaces the generator matrix **G** by an associated parity check matrix **H**.

Niederreiter encryption

- The private key contains **P**, **S** and **G** as for McEliece's cryptosystem.
- The public key is $\mathbf{H}_{pk} = \mathbf{SHP}$, for **H** the parity check matrix of C.
- The encryption of a message *msg* is:

  $$ctxt = \mathbf{H}_{pk} \times msg,$$

  msg being encoded as an error vector containing at most t ones.
- The decryption procedure computes:

  $$\mathbf{S}^{-1} \times ctxt = \mathbf{HP} \times msg,$$

  then a syndrome decoding algorithm followed by linear algebra is applied to recover *msg*.

Another difference is that in McEliece's scheme, an error is added to the encoded message, whereas here the message itself becomes the error. Niederreiter's scheme also provides some trade-offs in term of sizes and speed. Security-wise, both schemes are strictly equivalent. The Round 4 candidate Classic McEliece [ABC+20] is based on Niederreiter's scheme.

Other approaches

Note that code-based encryption schemes can be defined using different frameworks than the ones described above. The Round 4 candidates BIKE [ABB+20] and HQC [AAB+20] rely on different methodologies, for example the high-level structure of HQC is similar to the one of the selected standard Kyber, even if the underlying mathematical objects are very different.

3.3 Signatures

Obtaining secure and efficient signatures from error-correcting codes has been very hard to achieve so far. Several attempts have failed. We present here two secure but currently inefficient signature schemes, and two new proposals.

Hash-then-sign

In 2001, Courtois, Finiasz and Sendrier [CFS01] proposed a signature scheme based on the Hash-then-sign paradigm. It converts Niederreiter's scheme into a signature scheme: decryption becomes the signing procedure, and encryption becomes the verification procedure. Unfortunately, the parameters of [CFS01] do not scale well, which yields impractical parameters.

A recent construction [DST19] revisited the [CFS01] scheme and proposed to apply the GPV framework (see Section 2.3) to build code-based signatures. It is provably secure under a *new* code-based assumption.

Fiat-Shamir

Stern [Ste94, Ste96] and Véron [Vér96] proposed identification schemes from error-correcting codes. In this setting, the secret key is a vector $\mathbf{e}$, and the public key is a random matrix $\mathbf{H}$ as well as the syndrome $\mathbf{s} = \mathbf{H} \times \mathbf{e}$. The prover proves knowledge of $\mathbf{e}$ in a zero-knowledge way. Unfortunately, these protocols have soundness error between $2/3$ and $1/2$, so converting them into signature schemes via the Fiat-Shamir transform would require a few hundred repetitions. This would result in a slow signing procedure and large signatures.

A recent signature scheme, Durandal [ABG+19], uses partially *Fiat-Shamir with aborts* (see Section 2.3) in conjunction with the rank metric. It achieves a much better soundness than Stern-like protocols, but does not have a full security proof.

Summary

Inception:	1978
Assumptions:	Syndrome decoding, Distinguishing from a random code
Enc/KEM:	BIKE, Classic McEliece, HQC, NTS-KEM, ROLLO
Signatures:	CFS, Durandal, WAVE

4 Multivariate Cryptography

Multivariate cryptography is rather old since its inception dates back to 1988, when the first multivariate encryption scheme was proposed. It builds cryptosystems from problems involving multivariate polynomial equations over finite fields, for example the $\mathcal{MQ}$ problem.

However, over the years many schemes have been broken; while this may seem paradoxical, it is easily explained by the fact that these schemes relied on other, less secure and sometimes implicit assumptions. This has arguably undermined the credibility of the field.

The landscape of multivariate signatures is more mature than that of their encryption counterparts. There exist both hash-then-sign and Fiat-Shamir signatures with solid foundations, even though their concrete security for practical parameters can still be hard to pinpoint. As of July 2022, no multivariate scheme has been selected for standardization by NIST [NIS22].

Multivariate hash-then-sign schemes usually have small signatures (a few hundred bytes) at the expense of large keys. For Fiat-Shamir signatures, it is the other way around.

4.1 Hard problems

Multivariate cryptography is based on multivariate polynomial equations. For example:

$$f(x_1, x_2) = 3x_1^3x_2 + x_1^2 - x_2^3 + x_2 + 1$$

is a multivariate polynomial (of degree 4, since $x_1^3x_2$ is of degree 4). Solving problems involving multivariate polynomials of degree > 1 is conjectured hard for sufficiently large parameters. The variables are usually in a finite field (for example, $\mathbb{Z}_q$ for some prime q).

The $\mathcal{MQ}$ problem

The most famous problem is $\mathcal{MQ}$; it entails working with multivariate quadratics (i.e. multivariate polynomials of degree at most two), hence its name.

The $\mathcal{MQ}$ problem

Let $\mathbb{F}$ be a finite field. Let $\mathbf{F}(\mathbf{x})$ be $(f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$, where each $f_i : \mathbb{F}^n \to \mathbb{F}$ is a multivariate polynomial of degree at most 2 in $\mathbf{x} = (x_1, \dots, x_n)$. Let $\mathbf{y} \in \mathbb{F}^m$ and $\mathbf{F}$ be inputs to the problems below.

Decision: Is there $\mathbf{x}$ such that $\mathbf{F}(\mathbf{x}) = \mathbf{y}$?

Search: Find $\mathbf{x}$ such that $\mathbf{F}(\mathbf{x}) = \mathbf{y}$.

The decision version of $\mathcal{MQ}$ is NP-complete [GJ79], and its search version is NP-hard. This makes $\mathcal{MQ}$ an attractive option for building cryptographic schemes, but we will see that putting this idea into practice has been difficult to achieve.

The EIP problem

Except for Fiat-Shamir signatures, it is not known how to build cryptographic schemes solely on $\mathcal{MQ}$. Existing schemes rely, implicitly or explicitly, on at least a few other problems. The most prevalent is EIP, for *Extended Isomorphism of Polynomials* [DYC+08].

Extended Isomorphism of polynomials

Let $\mathcal{C}$ be a class of so-called quadratic *central maps* (from $\mathbb{F}^n$ to $\mathbb{F}^m$). Let $\mathbf{F} \in \mathcal{C}$ be a central map, and $\mathbf{S} : \mathbb{F}^n \to \mathbb{F}^n$ and $\mathbf{T} : \mathbb{F}^m \to \mathbb{F}^m$ be two affine maps.

Given $\mathbf{P} = \mathbf{S} \circ \mathbf{F} \circ \mathbf{T}$, find $\mathbf{F}$.

Many multivariate schemes rely on EIP, in the sense that solving EIP implies breaking the scheme. For most of these schemes, the person who generated **F** knows an efficient way of computing preimages for it. Typically, this results in **F** having some special structure.

This asymmetry between **F** (easy to invert) and **P** (hard to invert) makes it tempting to build public-key cryptography based on EIP: the public key would be **P** and the private key would contain **S**, **F** and **T**.

4.2 Encryption

The first multivariate encryption scheme, C^*, was introduced in 1988 by Matsumoto and Imai [MI88]. Like most multivariate encryptiom schemes, it relies at a high level on the ideas described in the paragraph about the EIP problem.

Multivariate encryption *à la* C^*

- The secret key consists of the maps $\mathbf{S}, \mathbf{F}$ and $\mathbf{T}$; all shall be easy to invert, and $\mathbf{S}, \mathbf{T}$ are affine maps.
- The public key is the map $\mathbf{P} = \mathbf{S} \circ \mathbf{F} \circ \mathbf{T}$, which is expected hard to invert.
- The encryption of a message *msg* is:

$$ctxt = \mathbf{P}(msg).$$

- The decryption of a ciphertext *ctxt* is:

$$\mathbf{T}^{-1} \circ \mathbf{F}^{-1} \circ \mathbf{S}^{-1}(ctxt).$$

At a high level, this mechanism is not too different from what is done in code-based encryption schemes such as McEliece or Niederreiter:

- The central map **F** plays the same role as the generator matrix **G**, as it allows to solve an otherwise untractable problem;
- The affine maps $\mathbf{S}, \mathbf{T}$ plays the same role as the permutation matrix **P** and invertible matrix **S** of McEliece, in the sense that they *hide* the structure of the central map.

While it may look simple to get public-key encryption from EIP using this blueprint, it has been notoriously hard to obtain secure schemes in practice. The original scheme by Matsumoto and Imai, as well as many subsequent schemes [TDTD13, PBD14, YS15], have been broken [Pat95, PPS17, CSV17], and only a handful of schemes remain unbroken. It is fair to say that building secure and efficient multivariate encryption schemes remains open.

4.3 Signatures

Multivariate signature schemes have been easier to obtain than encryption schemes. There exist schemes based both on the hash-then-sign paradigm, and the Fiat-Shamir paradigm.

4.3.1 Hash-then-sign

The high-level construction for multivariate Hash-then-sign schemes may be seen as the "dual" of the construction for encryption schemes, and is described in the next block. We can see that the signing and verification procedures mirror the decryption and encryption procedures of Section 4.2, respectively. This is very similar to how the code-based CFS signature mirrors Niederreiter's encryption scheme and, to a lesser extent, to how the RSA signature scheme mirrors the RSA encryption scheme.

Multivariate hash-then-sign

- The secret key are the maps **S**, **F** and **T**, which are easy to invert.
- The public key is the map $\mathbf{P} = \mathbf{S} \circ \mathbf{F} \circ \mathbf{T}$, expected hard to invert.
- The signature of a message *msg* is:
$$sig = \mathbf{T}^{-1} \circ \mathbf{F}^{-1} \circ \mathbf{S}^{-1} \circ H(msg).$$
- The verifier accepts a signature *sig* only if:
$$\mathbf{P}(sig) = H(msg).$$

The main difference among the schemes relying on this paradigm is the choice of the underlying field $\mathbb{F}$ and of the central map **F**. One of the more popular strategies is the *oil and vinegar* approach. This is for example the approach chosen by LUOV [BPSV19] and Rainbow [DCP+19]. Another popular approach relies on variants of *hidden field equations*, like GeMSS [CFM+17].

The public key **P** typically consists of m quadratic polynomials n variables over $\mathbb{F}$, so it is often large. On the other hand, each signature *sig* is essentially a vector in $\mathbb{F}^m$, and as a result signatures are often very small (a few hundred bytes).

Unfortunately, no multivariate hash-then-sign signature admits a security proof based on a standard assumption.

4.3.2 Fiat-Shamir with $\mathcal{MQ}$

The main idea of Fiat-Shamir signatures with $\mathcal{MQ}$ is that, for some publicly known map $\mathbf{F} : \mathbb{F}^m \rightarrow \mathbb{F}^n$, the secret key will be a vector $\mathbf{x} \in \mathbb{F}^m$, the public key will be $\mathbf{y} = \mathbf{F}(\mathbf{x}) \in \mathbb{F}^n$, and each signature will be a zero-knowledge proof that the signer knows **x**. An efficient way to do that was proposed by Sakumoto, Shirai and Hiwatari [SSH11]. Their key idea is to use the *polar* form of **F**, namely:

$$\mathbf{G}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{F}(\mathbf{x}_1 + \mathbf{x}_2) - \mathbf{F}(\mathbf{x}_1) - \mathbf{F}(\mathbf{x}_2).$$

A key property of **G** is that it is bilinear. [SSH11] leverage this property to split the private key in two, effectively enabling the construction of identification protocols based on the $\mathcal{MQ}$ problem. The authors proposed 3-pass (1 commitment, 1 challenge, 1 response) and 5-pass (2 commitments, 2 challenges, 1 response) identification protocols, but a conversion into a signature scheme was proposed only for the 3-pass protocol (via the Fiat-Shamir transform).

Another step in the direction of provably secure signatures from $\mathcal{MQ}$ was done by Chen, Hülsing, Rijneveld, Samardjiska and Schwabe [CHR+16]. As the signatures obtained from the 3-pass protocol from [SSH11] were too inefficient, [CHR+16] instead adapted the 5-pass identification protocol so that it could be converted into a signature scheme, once again via the Fiat-Shamir transform. The resulting scheme, MQDSS, is proven secure under the $\mathcal{MQ}$ assumption. The scheme SOFIA [CHR+18] uses the same ideas but with a proof against adversaries with stronger quantum capabilities.

For such schemes, the public and secret keys are usually very small since they are the vectors **x** and **y**, respectively. However, the underlying identification protocol only achieves a soundness of about $\frac{1}{2}$, thus the signing procedure requires to repeat this protocol several times. As a result the signatures are rather large (more than 30 kB).

Summary

Inception:	1988
Assumptions:	$\mathcal{MQ}$, EIP
Enc/KEM:	-
Signatures:	LUOV, MQDSS, Rainbow, GeMSS

5 Signatures from One-Way Functions

One-way functions are functions which are easy to compute, but hard to invert; for example, the hash functions SHA-2 and SHA-3 are assumed to be one-way functions.

The idea to build signatures from one-way functions was first proposed independently by Lamport [Lam79] and Merkle [Mer90] in 1979 (the work of Merkle was published 10 years later). Until recently, all the signatures based on one-way functions could be seen as loosely related to the hash-then-sign paradigm, but signatures relying on the Fiat-Shamir paradigm were recently proposed, and their philosophy is very different.

From a security viewpoint, these signatures are very appealing. Indeed, it is proven [Rom90] that signatures exist if and only if one-way functions exist. Consequently, in terms of underlying hypotheses, signatures based on one-way functions are the best one could possibly hope for. In addition, all existing constructions are provably secure.

Unfortunately, these perks come at the cost of efficiency; all existing schemes have slow signing procedures as well as large signatures. Some of these schemes achieve higher efficiency, but require in exchange to maintain an internal *state*; but this is not always possible, as some deployment contexts preclude this possibility. However, if one accepts these restrictions, these signatures provide strong security guarantees.

There also is one method for building an encryption scheme from minimal assumptions, however it is extremely inefficient [Mer78], and this seems to be inherent [BM09].

5.1 Hard problems

Informally, a one-way function is a function easy to compute but hard to invert. One can define several hardness assumptions for one-way functions and related notions. The most common assumptions are preimage, second preimage and collision resistance. Preimage resistance of a function H essentially states that H is hard to invert for a specified output y. Second preimage resistance makes a slightly different statement.

The preimage problem

Let $H : X \to Y$ be a function, and $y \in Y$.
Find $x \in X$ such that $H(x) = y$.

The second preimage problem

Let $H : X \to Y$ be a function, and $x_1 \in X$.
Find $x_2 \neq x_1$ such that $H(x_1) = H(x_2)$.

Collision resistance states it is hard to find two inputs that H maps to the same output.

The collision problem

Let $H : X \to Y$ be a function.
Find $x_1 \neq x_2$ such that $H(x_1) = H(x_2)$.

Preimage vs second preimage vs collision

It is possible to build functions which are hard for any of these problems and easy for any other one (with the exception that collision resistance always implies second preimage resistance). However, these examples are quite contrived and mostly of theoretical interest.

The best classical attacks known for finding preimages on a generic function $H : \{0,1\}^* \to \{0,1\}^n$ require about 2^n operations, compared to $2^{n/2}$ operations for finding collisions. In practice, there exist hash functions for which collisions have been found, but no (second-) preimage attack has been mounted: this includes MD5 [dB94] and SHA-1 [SBK+17].

The situation of quantum attacks is less clear as it is a recent field. Grover's algorithm [Gro96] reduces the quantum cost of preimage search to $O(2^{n/2})$. Quantum speed-ups have been proposed for collision as well [BHT98, CNS17], their practical applicability is still being discussed.

5.2 Hash-based signatures

Hash-based signatures are based on the observation that a hash function H allows to commit to a secret key, while hiding it. In this perspective, the public key will be a commitment of the private key.

Signing a message typically consists in revealing partial information so that the verifier can recompute the commitment and check it against the public key. A peculiar (but useful) property of hash-based signatures in general is that one can recover the public key from a valid signature and the associated message.

One-time signatures

One-time signature (toy example)

- The private key is a bitstring $sk = (sk_1, sk_2)$.
- The public key is $pk = (H^M(sk_1), H^M(sk_2))$, H^M denoting the M-times iteration of H
- The signature of a message msg in $\{0, \ldots, M\}$ is:

$$(sig_1, sig_2) = (H^{msg}(sk_1), H^{M-msg}(sk_2)).$$

- The verifier accepts the signature if and only if:

$$(H^{M-msg}(sig_1), H^{msg}(sig_2)) = pk.$$

The scheme described in the "One-Time Signature (Toy Example)" box puts into practice the high-level ideas described at the beginning ot this section.

Without knowing sk, the only way to forge a signature for msg is to invert H. Therefore, this scheme is secure under the preimage hardness of H. However, one can also show that given signatures for two different messages $msg_1 \neq msg_2$, one can compute a signature for any message $msg_1 < msg_3 < msg_2$. Therefore, this scheme is secure if it signs at most one message. These kind of schemes are called *one-time* signatures (or OTS), and this is obviously a huge limitation.

From one-time to few-times: Merkle trees

One-time signatures can be converted to few-time signatures (which means that a few messages can be signed) by using *Merkle trees* [Mer90], illustrated in Figure 4.

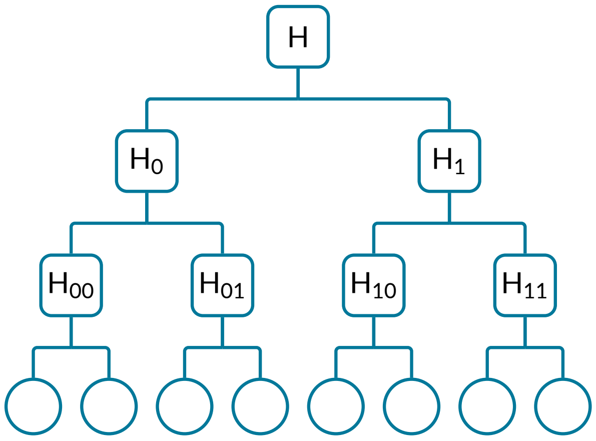


Figure 4: A Merkle tree

In a Merkle tree, each internal node (square nodes in Figure 4) is the hash of the concatenation of its children: this is illustrated by the edges linking these nodes to their children. For signatures, the leaves of a Merkle tree (circular nodes in Figure 4) are the public keys of OTS. The root of a Merkle tree (its top node) is a commitment of all the public keys (thus of the private keys as well).

Merkle signature

- The private key is the set of all OTS private keys: $sk = \{sk_{000}, \ldots, sk_{111}\}$;
- The public key is the root of the Merkle tree: $pk = \mathsf{H}$;
- The signer uses a leaf OTS to sign *msg*, and sends the one-time signature along with nodes which allow to recover the root of the Merkle tree. For example, a valid signature may be:

$$sig = (sig_{000}, pk_{001}, \mathsf{H}_{01}, \mathsf{H}_1)\ ,$$

  where sig_{000} is a signature of *msg* using sk_{000} (the private key associated to pk_{000}).
- The verifier uses the key-recovery property of hash-based signatures to compute a public key from the one-time signature (here, he gets pk_{000} from sig_{000}), and recovers the top root of the Merkle tree from *sig*. For the example given above:

$$H(H(H(pk_{000} \| pk_{001}) \| \mathsf{H}_{01}) \| \mathsf{H}_1) = \mathsf{H} = pk.$$

  If it succeeds, the signature is accepted. Otherwise, it means that a part of the signature is incorrect, and it is rejected.

The Merkle tree in Figure 4 can sign up to 8 messages, which is better than one but is still far from perfect. Another caveat is that it still requires to keep track of the OTS keys used; indeed, once an OTS key is used, it can by definition not be reused. The "*bookkeeping*" that this method imposes on the signer is called *statefulness*, and it is an risky property that can be difficult to enforce, especially in distributed systems.

Going stateless

Several other techniques allow to improve the efficiency and flexibility of hash-based signatures, and so we only briefly mention them here. Goldreich trees are a flexible variant of Merkle trees, which allow to relax the statefulness requirement to some extent. Stateless few-time signatures [RR02, updated version] also provide some extra flexibility. Hash-based signatures can be organized in two families:

- *Stateful* signatures still require to maintain a state. However, some of them attain reasonable signature sizes (less than 3 kB), such as LMS [LM95] and XMSS [HBG+18], which have recently been standardized by NIST [NIS20].
- *Stateless* signatures manage, by increasing parameters, to avoid the need for a state management. This comes at the cost of reduced efficiency; for example, the SPHINCS+ scheme yields signatures of about 30 kB.

Both families of schemes share several common points. For example, the public key is very small (around 64 bytes), because it is a single hash. Also, the signing procedure entails a large number of hash computations, and is therefore rather slow. Finally, because of the tree structure, these schemes only support a limited number of hashes; however, this number can be made arbitrarily large by setting the parameters adequately (SPHINCS+ supports up to 2^{64} messages).

We note that the stateless hash-based signature scheme SPHINCS+ [HBD+20] has been selected by NIST for standardization in July 2022 [NIS22].

5.3 Zero-knowledge signatures

In 2017, Chase et al. [CDG+17] proposed a novel way of using one-way functions to yield post-quantum signature schemes. Their work relies on the Fiat-Shamir transform, but also on other ideas, such as *secret sharing* and *multiparty computation*.

First proposed by Shamir [Sha79] and Blakley [Bla79], secret sharing consists of splitting a secret into *shares*, in a way that the secret can only be retrieved by combining sufficiently many shares. For example, given a binary value $b \in \{0, 1\}$, if we split b as $b = b_1 \oplus b_2$, where b_1 is uniformly random, then b_1 or b_2 alone do not provide any information about b, but knowing both allows to recover b.

Multiparty computation (or MPC for short) [Yao82, Yao86, GMW87], proposes solutions to perform computation over shared data, while preserving the purpose of secret sharing (that is, many shares must be combined to recover the secret). Multiparty computation has found many applications, like distributed computation, protection against side-channel attacks, etc.

MPC-in-the-head [IKOS07] performs an MPC computation and reveals intermediate data (though not enough to reveal any secret) in a pseudorandom manner. For example, given y, if one wants to prove that they know x such that $H(x) = y$, they can perform an MPC-in-the-head computation of $H(x)$ with, say, 3 shares, and send a transcript. This will convince a verifier (up to probability $1/3$) that the prover knows x. At a high level, this is not too different from how the Fiat-Shamir transform renders an identification protocol non-interactive.

[CDG+17] builds a signature scheme based on this idea (and others). This approach is quite different from hash-based signatures; these view a one-way function H as a black box, whereas the internal description of H is quite relevant for [CDG+17]. As each MPC-in-the-head transcript convinces the verifier with probability $1/3$, it needs to be repeated several times (200 to 800 in practice), which results in a slow signing procedure and large signatures. The resulting scheme, however, is as secure as the underlying function H. Picnic [ZCD+17] is an application of this idea.

Summary

Inception:	1979
Assumptions:	Collision or (second) preimage resistance of one-way functions
Enc/KEM:	-
Signatures:	XMSS, SPHINCS$^+$, Picnic

6 Isogeny-based Cryptography

Isogeny-based cryptography is the youngest of the subfields of post-quantum cryptography studied in this document, since it really started in 2006.

For key-exchange and encryption, the idea is mostly to *revisit* the classical elliptic curve Diffie-Hellman and El-Gamal schemes, except that instead of working with points of elliptic curves, the elliptic curves themselves become the objects which are manipulated, and this is done through the use of *isogenies*, which are a class of maps between elliptic curves.

Signature schemes based on the Fiat-Shamir transform have been proposed recently.

For both key-establishment and signatures, this transposition has not been straightforward; the schemes are often not as "natural" as their elliptic curve counterparts, and efficiency issues are still being addressed at this time. While they are currently slow, these schemes offer excellent communication costs compared to schemes of other families.

This is a recent field, so the security estimates, parameters and efficiency of schemes are likely to evolve. The Round 4 candidate SIKE [JAC+20] is based on isogenies.

6.1 Hard problems

We recall that an elliptic curve is the set of points (x, y) that, for fixed (a, b), verify:

$$y^2 = x^3 + ax + b,$$

with a, b, x, y belonging to (the algebraic closure of) a finite field. Isogeny-based cryptography can work on two classes of elliptic curves, *ordinary* or *supersingular*. The second class currently seems to provide the best efficiency/security trade-off, and almost all schemes use it.

The isogeny problem

For the purpose of this document, it is sufficient to remember that isogenies are a specific class of maps between elliptic curves, and that isogenous curves are curves which are connected by an isogeny.

The isogeny problem

Given two supersingular isogenous curves E_1, E_2, compute an isogeny:

$$\varphi : E_1 \to E_2.$$

At a high level, we can see that this is similar to the discrete logarithm problem: instead of looking for a such that $g^a = h$, we look for an isogeny φ mapping E_1 to E_2.

6.2 Key-exchange

The idea of using isogenies to replicate the Diffie-Hellman protocol was first proposed by Couveignes [Cou06], and rediscovered by Rostovtsev and Stolbunov [RS06],

6.2.1 The Couveignes-Rostovtsev-Stolbunov scheme (CRS)

We recall that classical Diffie-Hellman starts with a public element g. Alice sends g^a, Bob sends g^b and at the end of the key-exchange they both have a shared secret g^{ab}. The Couveignes-Rostovtsev-Stolbunov scheme (or CRS) can be seen as a generalization of this idea, and is described in the next diagram, with the public element E at the top left, and the shared secret $E/\langle P, Q\rangle$ at the bottom right.

$$\begin{tikzcd}
E \arrow[r, "\phi"] \arrow[d, "\psi"'] & E_\phi \arrow[d, "\psi"] \\
E_\psi \arrow[r, "\phi"] & E_{\psi,\phi}
\end{tikzcd}$$

While classical (elliptic-curve) Diffie-Hellman takes g to be an element of an elliptic curve E, the CRS protocol subsitutes g with the elliptic curve E, and the action of exponentiating a point is replaced by an isogeny mapping E to another elliptic curve. Both Alice and Bob keep their isogenies (ϕ and ψ) secret, but at the end of the protocol they both share a known secret $E_{\psi,\phi}$.

Later, [CJS14] proposed quantum algorithms for computing isogenies in the CRS setting. To account for these algorithms, CRS required a growth in parameters which made it impractical at the time.

6.2.2 SIDH

Jao and de Feo [JD11] proposed an analog of the Diffie-Hellman key exchange over supersingular curves.

$$\begin{tikzcd}
E \arrow[r, "\phi_A"] \arrow[d, "\phi_B"'] & E/\langle H_A \rangle \arrow[d, "{\phi_B,\{\phi_A(P_B),\phi_A(Q_B)\}}"] \\
E/\langle H_B \rangle \arrow[r, "{\phi_A,\{\phi_B(P_A),\phi_B(Q_A)\}}"] & E/\langle H_A, H_B \rangle
\end{tikzcd}$$

This paradigm of key-exchange is generally called *Supersingular Isogeny Diffie-Hellman*, or SIDH. From a security perspective, this variant is somewhat incomparable to the ordinary case proposal from [Cou06, RS06]. On one hand, the [CJS14] algorithm does not apply in this case. On the other hand, the adversary has access to some additional information compared to [Cou06, RS06]. Therefore, the security of this scheme depends on a different problem: the SIDH problem.

The SIDH problem

Given two supersingular isogenous curves E_1 and E_2, compute an isogeny

$$\varphi : E_1 \to E_2,$$

knowing $\varphi(P)$ and $\varphi(Q)$ for some precise P, Q (we omit the details here).

From a practical perspective, being able to work on supersingular curves allows much smaller parameters, which results in more compact and more efficient schemes. The Round 4 candidate SIKE [JAC+20] is based on SIDH.

In July 2022, a devastating attack by Castryck and Decru [CD22] has broken the SIDH scheme and SIKE. Crucially, it exploits the additional information $\varphi(P)$ and $\varphi(Q)$. As of August 2022, it is unclear whether the attack can be mitigated.

6.2.3 Non-interactive key-exchange: CSIDH and revisited CRS

A nice property which the original Diffie-Hellman protocol relies on is *commutativity*:

$$(g^a)^b = (g^b)^a = g^{ab}.$$

The main issues with SIDH (somewhat ad-hoc problem and need for an interactive protocol) stem from the non-commutativity of the underlying operation. Therefore, very recent works have tried enforcing a commutative operation.

The first work, by [DKS18], revisited the CRS scheme, which already possessed this commutativity property but was impractical. The work of [DKS18] proposed mainly algorithmic improvements, and yields key-exchange on *ordinary* curves. It is still very slow.

On the other hand, [CLM+18] showed that, by restricting to subsets of the elliptic curves and of the isogenies, one could effectively construct a commutative group action over *supersingular* curves. Combining this with the algorithmic improvements of [DKS18] allowed them to adapt the CRS scheme over supersingular curves. The resulting scheme is called CSIDH. It remains rather slow, but small public key sizes make it attractive.

As a direct consequence of the commutativity of their underlying group actions, these two schemes are non-interactive, support static keys and are directly protected against active attacks (they do not require a CCA transform). The first two properties are desirable for several real-life applications, and the third one makes the schemes faster and simpler to implement. A caveat to both schemes (besides efficiency) is that the best quantum attacks are subexponential. In particular, recent works [BS20, Pei20, CCJR20] seem to imply that CSIDH requires much larger parameters than initially expected.

6.3 Signatures

While the first identification schemes (which can often be converted in signatures) have been proposed as early as 2014 [FJP14], concrete signature schemes based on isogenies have only appeared recently. One can expect significant improvements of the schemes and cryptanalysis in the future.

6.3.1 Early schemes

In 2017, two papers [YAJ+17, GPS17] proposed signature schemes based on supersingular isogenies. The first scheme is in both papers, and follows from the identification scheme from [FJP14]. The second one, proposed only in [GPS17], relied on more recent algorithmic techniques by [KLPT14].

The idea of [GPS17] is that, given three elliptic curves E, E_{com}, E_{pk} and two isogenies $\phi_{sk} : E \to E_{pk}$ and $\phi_0 : E \to E_{com}$, one can use the techniques of [KLPT14] to compute an isogeny $\phi_1 : E_{com} \to E_{pk}$, which is summarized below.

$$\begin{tikzcd}
E \arrow[r, "\phi_0"] \arrow[dr, "\phi_{sk}"'] & E_{com} \arrow[d, dashed, "\phi_1"] \\
 & E_{pk}
\end{tikzcd}$$

We now describe a simplified version of the signature scheme from [GPS17].

GPS protocol

- The public key consists of two curves E and E_{pk};
- The private key is an isogeny $\phi_{sk} : E \to E_{pk}$. This is done by first choosing E and ϕ_{sk} randomly and then computing E_{pk};
- A round of the protocol is as follows:
  - **Commit:** Choose a random isogeny ϕ_0 and compute $E_{com} = \phi_0(E)$;
  - **Challenge:** The challenge c is 0 or 1.
  - **Response:** If $c = 0$, then $rsp = (E_{com}, \phi_0)$. Else, $rsp = (E_{com}, \phi_1)$, where $\phi_1 : E_{com} \to E_{pk}$ is computed using [KLPT14].

  Return *rsp*.
- The verifier accepts $rsp = (E_{com}, \varphi_c)$ if and only if either:
  - $c = 0$ and φ_0 sends E to E_{com}.
  - $c = 1$ and φ_1 sends E_{com} to E_{pk}.

  φ_c should of course be an isogeny.

At a high level, the underlying identification protocol resembles the well-known proof system of [GMW86] for proving knowledge of an isomorphism between graphs. Both protocols have a soundness of $\frac{1}{2}$, which informally means that an illegitimate prover may falsely convince a verifier that he knows the secret key, with a probability $\frac{1}{2}$.

To reach cryptographic levels of confidence, between 128 and 256 repetitions are needed. This yields rather large signatures.

6.3.2 SeaSign and CSI-FiSh

Recently, de Feo and Galbraith [FG19] have taken advantage of the commutative group action introduced in [CLM+18] to propose a shorter signature scheme.

An interesting idea of [FG19] is to minimize soundness error by having a large number n of curves $E_{pk}^{(i)}$ composing the public key, then using Merkle trees to prevent an explosion of the public key size. Interestingly, the scheme also relies on techniques first developed for lattice-based cryptography: the identification protocol uses the aborting techniques from [Lyu09], and this protocol is converted into a secure signature scheme using recent work from [KLS18] originally targeting lattice-based schemes. The resulting scheme is called SeaSign. As for [GPS17], the signature consists of proving the knowledge of an isogeny between two curves.

Beullens, Kleinjung and Vercauteren [BKV19] proposed several efficiency improvements to SeaSign. In particular, they compute an efficient representation of a class group (an algebraic structure) underlying much of the computations in SeaSign. This efficient representation, along with other tricks, enables us to substantially improve the efficiency of the scheme. The resulting scheme, CSI-FiSh, is extremely compact. When optimizing for signature size, the size can be as small as 263 bytes for 128 bits of classical security; when optimizing for combined public key and signature sizes, the total size can be as small as 1468 bytes. However, it is also currently very slow. Note that these numbers only apply for the parameter set CSI-FiSh-512, the security of which is under scrutiny [BS20, Pei20, CCJR20] and might call for updated parameters.

6.3.3 SQISign

Very recently, De Feo, Kohel, Leroux, Petit and Wesolowski [DKL+20] proposed a new declination of the [GPS17] scheme. A notable improvement compared to previous schemes is that the base protocol has a very small soundness error (compared to the $1/2$ in [GPS17] or $1/(n+1)$ in SeaSign). Consequently, one single round of the protocol is sufficient and the resulting signature scheme is therefore extremely compact.

$$\begin{tikzcd}
E \arrow[r, "\phi_{com}"] \arrow[d, "\phi_{sk}"'] & E_{com} \arrow[d, "\psi_{chal}"] \\
E_{pk} \arrow[r, "\phi_{rsp}"] & E_{chal}
\end{tikzcd}$$

The principle of SQISign is illustrated above, and explained below.

- The public key is comprised of two elliptic curves E and E_{pk}.
- The private key is an isogeny ϕ_{sk} sending E to E_{pk}.
- A signature simulates a commit-challenge-response identification protocol, where:
  - The commitment is E_{com}.
  - The challenge E_{chal} is computed by applying an isogeny ψ_{chal} to E_{com}.
  - The response is an isogeny ϕ_{rsp} sending E_{pk} to E_{chal}.

Summary

Inception:	2006
Hard Problems:	Isogeny Problem, SIDH Problem, CSIDH Problem
Enc/KEM:	SIKE, CSIDH
Signatures:	CSI-FiSh [BKV19], SQISign [DKL+20]

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